Conservation equation on braneworlds in six dimensions

Georgios Kofinas*
Departament de Física Fonamental, Universitat de Barcelona, Diagonal 647, 08028 Barcelona, Spain

(Dated: February 7, 2008)

We study braneworlds in six-dimensional Einstein-Gauss-Bonnet gravity. The Gauss-Bonnet term is crucial for the equations to be well-posed in six dimensions when non-trivial matter on the brane is included (the also involved induced gravity term is not significant for their structure), and the matching conditions of the braneworld are known. We show that the energy-momentum of the brane is always conserved, independently of any regular bulk energy-momentum tensor, contrary to the situation of the five-dimensional case.

Much work on braneworlds in six-dimensional spacetimes has been done, especially during the last two years [1]. It is known that six-dimensional Einstein gravity cannot support a (thin) braneworld with a non-trivial matter content different than a brane tension [2]. The situation can be improved if a Gauss-Bonnet term is added in the bulk action, in which case the generic matching conditions of a 3-brane were derived in [3]. In the present note, we show that in such bulks the energy momentum tensor of the brane is always conserved, independently of the bulk energy-momentum tensor. On the contrary, in five-dimensional bulks, as known, the conservation of energy-momentum on the brane is violated when there is energy transfer between brane and bulk, due to non-vanishing transverse-parallel components of the bulk energy-momentum tensor.

We consider the total gravitational brane-bulk action

$$S_{gr}=\frac{1}{2\kappa_6^2}\int d^6x\sqrt{-|\mathsf{g}|}\,\Big\{\mathcal{R}-2\Lambda_6+\alpha\Big(\mathcal{R}^2-4\mathcal{R}_{AB}\,\mathcal{R}^{AB}+\mathcal{R}_{ABCD}\,\mathcal{R}^{ABCD}\Big)\Big\}+\frac{r_c^2}{2\kappa_6^2}\int d^4x\sqrt{-|g|}\,(R-2\Lambda_4), \quad (1)$$

where calligraphic quantities refer to the bulk metric tensor g, while the regular ones to the brane metric tensor g. The Gauss-Bonnet coupling α has dimensions $(length)^2$ and is defined as

$$\alpha=\frac{1}{8g_s^2}\,, \quad (2)$$

with g_s the string energy scale, while from the induced-gravity crossover lenght scale r_c we can define

$$r_c=\frac{\kappa_6}{\kappa_4}=\frac{M_4}{M_6^2}\,. \quad (3)$$

Here, M_6 is the fundamental six-dimensional Planck mass $M_6^{-4}=\kappa_6^2=8\pi G_6$, while M_4 is given by $M_4^{-2}=\kappa_4^2=8\pi G_4$. The brane tension is

$$\lambda=\frac{\Lambda_4}{\kappa_4^2}\,. \quad (4)$$

The field equations arising from the action (1) are

$$\mathcal{G}_{AB}-\frac{\alpha}{2}(\mathcal{R}^2-4\mathcal{R}_{CD}\mathcal{R}^{CD}+\mathcal{R}_{CDEF}\mathcal{R}^{CDEF})\mathsf{g}_{AB}+2\alpha\times(\mathcal{R}\mathcal{R}_{AB}-2\mathcal{R}_{AC}\mathcal{R}_B{}^C-2\mathcal{R}_{ACBD}\mathcal{R}^{CD}+\mathcal{R}_{ACDE}\mathcal{R}_B{}^{CDE})=\kappa_6^2\mathcal{T}_{AB}-\Lambda_6\mathsf{g}_{AB}+\kappa_6^2\,{}^{(loc)}T_{AB}\,\delta^{(2)}, \quad (5)$$

where $\mathcal{T}_{AB}$ is a regular bulk energy-momentum tensor, T_{AB} is the brane energy-momentum tensor, ${}^{(loc)}T_{AB}=T_{AB}-\lambda g_{AB}-(r_c^2/\kappa_6^2)G_{AB}$, and $\delta^{(2)}$ is the two-dimensional delta function. Capital indices $A,B,...$ are six-dimensional. Assuming that the bulk metric in the brane-adapted coordinate system takes the axially symmetric form

$$ds_6^2=dr^2+L^2(x,r)d\varphi^2+g_{\mu\nu}(x,r)dx^\mu dx^\nu, \quad (6)$$

with $g_{\mu\nu}(x,0)$ being the braneworld metric and φ having the standard periodicity 2π, under the usual assumptions for conical singularities $L(x,r)=\beta(x)r+\mathcal{O}(r^2)$ for $r\approx 0$, $\partial_rL(x,0)=1$, $\partial_rg_{\mu\nu}(x,0)=0$, the general matching conditions for imbedding the 3-brane in the six-dimensional theory (1) were found in [3] as follows

$$K^{\alpha\lambda}{}_\lambda K_{\alpha\mu\nu}-K^{\alpha\lambda}{}_\mu K_{\alpha\nu\lambda}+\frac{1}{2}(K^{\alpha\lambda\sigma}K_{\alpha\lambda\sigma}-K^{\alpha\lambda}{}_\lambda K_\alpha{}^\sigma{}_\sigma)g_{\mu\nu}+\Big(\beta^{-1}-1+\frac{r_c^2}{8\pi\alpha\beta}\Big)G_{\mu\nu}+\frac{\kappa_6^2\lambda-2\pi(1-\beta)}{8\pi\alpha\beta}g_{\mu\nu}=\frac{\kappa_6^2}{8\pi\alpha\beta}T_{\mu\nu}. \quad (7)$$

Here, $K_{\alpha\mu\nu}=\mathsf{g}(\nabla_\mu n_\alpha,\partial_\nu)=n_{\alpha\mu;\nu}$ denote the extrinsic curvatures of the brane (symmetric in μ,ν), where n_α ($\alpha=1,2$) are arbitrary unit normals to the brane (indices $\alpha,\beta,...$ are lowered/raised with the matrix $\mathsf{g}_{\alpha\beta}=\mathsf{g}(n_\alpha,n_\beta)$ and its inverse $\mathsf{g}^{\alpha\beta}$), while ∇ (also denoted by ;) refers to the Christoffel connection of g. For extracting this singular part of equations (5), one has to focus on the worst behaving pieces with the structure $\delta(r)/L\sim\delta(r)/r$. Note that with respect to local rotations $n_\alpha\to O_\alpha{}^\beta(x^A)\,n_\beta$, $K_{\alpha\mu\nu}\to O_\alpha{}^\beta K_{\beta\mu\nu}$ transforming as a vector, thus Eq.(7) is invariant under changes of the normal frame. It is also noticeable that the matching conditions (7) are quadratic in the extrinsic curvature, while the corresponding matching conditions of 5-dimensional Gauss-Bonnet theory are cubic [4]. Focusing on the $\mathcal{O}(1/r)$ terms in the $r\mu$ components of equations

*gkofin@phys.uoa.gr; kofinas@ffn.ub.es

(5) we obtain the equation (correcting equation (16) of [3])

$$\mathcal{R}^{\alpha\sigma}{}_{\nu\sigma}K_\alpha{}^\lambda{}_\lambda-\mathcal{R}^{\alpha\sigma}{}_{\lambda\sigma}K_\alpha{}^\lambda{}_\nu-\mathcal{R}^{\alpha\lambda}{}_{\nu\sigma}K_\alpha{}^\sigma{}_\lambda=\frac{\beta_{,\mu}}{\beta}\Big[G^\mu_\nu-\frac{1}{4\alpha}\delta^\mu_\nu$$
$$+K^{\alpha\sigma}{}_\nu K_\alpha{}^\mu{}_\sigma-K^{\alpha\sigma}{}_\sigma K_\alpha{}^\mu{}_\nu+\frac{1}{2}(K^{\alpha\sigma}{}_\sigma K_\alpha{}^\lambda{}_\lambda-K^{\alpha\sigma\lambda}K_{\alpha\sigma\lambda})\delta^\mu_\nu\Big]. \tag{8}$$

In higher codimensions, one has geometrical equations analogous to those holding for hypersurfaces [5, 6], namely

$$\mathcal{R}_{\mu\nu\kappa\lambda}=R_{\mu\nu\kappa\lambda}+K^\alpha{}_{\mu\lambda}K_{\alpha\nu\kappa}-K^\alpha{}_{\mu\kappa}K_{\alpha\nu\lambda}\,, \tag{9}$$

$$\mathcal{R}^\alpha{}_{\mu\nu\lambda}=K^\alpha{}_{\mu\nu!\lambda}-K^\alpha{}_{\mu\lambda!\nu}\,, \tag{10}$$

$$\mathcal{R}^\beta{}_{\alpha\mu\nu}=\Omega^\beta{}_{\alpha\mu\nu}+K_{\alpha\mu}{}^\lambda K^\beta{}_{\nu\lambda}-K_{\alpha\nu}{}^\lambda K^\beta{}_{\mu\lambda}\,. \tag{11}$$

The covariant derivative ! is defined with respect to the connection $\varpi_{\beta\alpha\mu}=\mathsf{g}(\nabla_\mu n_\alpha,n_\beta)$ as

$$\Phi^\alpha_{\beta!\mu}=\Phi^\alpha_{\beta|\mu}+\varpi^\alpha{}_{\gamma\mu}\Phi^\gamma_\beta-\varpi^\gamma{}_{\beta\mu}\Phi^\alpha_\gamma, \tag{12}$$

for fields Φ^α_β transforming as tensors under normal frame rotations, $\Phi^\alpha_\beta\rightarrow O_\beta{}^\delta(O^{-1})_\gamma{}^\alpha\Phi^\gamma_\delta$, and | refers to the Christoffel connection $\gamma_{\mu\nu\lambda}=\mathsf{g}(\nabla_\lambda\partial_\nu,\partial_\mu)$ of the induced brane metric $g_{\mu\nu}$ [the derivative | in (12) is meant on tangential indices $\mu,\nu,...$ that Φ^α_β may possess]. $\Omega^\beta{}_{\alpha\mu\nu}$ is the curvature of the connection $\varpi^\beta{}_{\alpha\mu}$, $\Omega^\beta{}_{\alpha\mu\nu}=2\varpi^\beta{}_{\alpha[\nu,\mu]}+2\varpi^\gamma{}_{\alpha[\nu}\varpi^\beta{}_{|\gamma|\mu]}$ [7].

From equations (7), differentiating with respect to ! and making use of the identity (10), we obtain

$$\frac{\kappa_6^2}{8\pi\alpha\beta}T^\mu_{\nu|\mu}=\mathcal{R}^{\alpha\mu}{}_{\nu\mu}K_\alpha{}^\lambda{}_\lambda-\mathcal{R}^{\alpha\mu}{}_{\lambda\mu}K_\alpha{}^\lambda{}_\nu-\mathcal{R}^{\alpha\lambda}{}_{\nu\mu}K_\alpha{}^\mu{}_\lambda$$
$$+\frac{\beta_{,\mu}}{\beta^2}\Big[\frac{\kappa_6^2}{8\pi\alpha}T^\mu_\nu+\frac{2\pi-\kappa_6^2\lambda}{8\pi\alpha}\delta^\mu_\nu-\Big(1+\frac{r_c^2}{8\pi\alpha}\Big)G^\mu_\nu\Big]. \tag{13}$$

It is noticeable that equation (13) can also arise from equations (7) and (9) without using (10). Indeed, if $S_{\mu\nu}=K^{\alpha\lambda}{}_\lambda K_{\alpha\mu\nu}-K^{\alpha\lambda}{}_\mu K_{\alpha\nu\lambda}$, the matching conditions (7) take the form

$$S_{\mu\nu}-\frac{1}{2}Sg_{\mu\nu}=c_1G_{\mu\nu}+c_2g_{\mu\nu}+c_3T_{\mu\nu}, \tag{14}$$

($S=S_{\mu\nu}g^{\mu\nu}$) with c's being the appropriate coefficients, while the contraction of the generalized Gauss-Codazzi equation (9) gives

$$S_{\mu\nu}=R_{\mu\nu}-\mathcal{R}^\lambda{}_{\mu\lambda\nu}. \tag{15}$$

From equations (14), (15) we get

$$\mathcal{R}^{\lambda\mu}{}_{\lambda\nu}-\frac{1}{2}\mathcal{R}^{\lambda\sigma}{}_{\lambda\sigma}\delta^\mu_\nu=(1-c_1)G^\mu_\nu-c_2\delta^\mu_\nu-c_3T^\mu_\nu. \tag{16}$$

Considering the μ-covariant derivative with respect to | of equation (16), transforming the | derivatives on the left hand side into ; derivatives, and making use of the Bianchi identity, we arrive at equation (13). Note that in equation (16) the extrinsic curvatures have been eliminated, and the effective geometry is directly connected to the bulk geometry. Finally, making use of equations (7), (8), (13), we obtain the exact conservation equation on the brane

$$T^\mu_{\nu|\mu}=0, \tag{17}$$

independently of the energy momentum tensor in the bulk. This is contrary to the five-dimensional case, where equation (17) is valid only if the mixed transverse-parallel components of the bulk energy-momentum tensor vanish.

In conclusion, we have considered a codimension two (thin) braneworld in Einstein-Gauss-Bonnet (-induced gravity) theory, where the addition of the Gauss-Bonnet term is known to make meaningful the situation when non-trivial braneworld matter content is included. Using appropriate components of the field equations, and the generalized Gauss-Codazzi-Mainardi equations holding for higher codimensions, we show that the energy-momentum of the brane is always conserved, independently of the bulk energy momentum tensor, or a possibly variable deficit angle.

Acknowlegements We wish to thank C. Charmousis and J. Garriga for useful discussions. This work is supported by a European Commission Marie Curie Fellowship, under contract HMPF-CT-2004-400737.

[1] S. Deser, R. Jackiw and G. 't Hooft, Ann. Phys. **152**, 220 (1984); R. Geroch and J. Traschen, Phys. Rev. **D36**, 1017 (1987); R. Gregory, Phys. Rev. **D54**, 4955 (1996); U. Gunther and A. Zhuk, Phys. Rev. **D56**, 6391 (1997) [gr-qc/9706050]; R. Sundrum, Phys. Rev. **D59**, 085010 (1999) [hep-ph/9807348]; A. Cohen and D. Kaplan, Phys. Lett. **B470**, 52 (1999) [hep-th/9910132]; R. Gregory, Phys. Rev. Lett. **84**, 2564 (2000) [hep-th/9911015]; U. Gunther and A. Zhuk, Phys. Rev. **D61**, 124001 (2000) [hep-ph/0002009]; J-W. Chen, M. Luty and E. Ponton, JHEP **0009**, 012 (2000) [hep-th/0003067]; T. Gherghetta and M. Shaposhnikov, Phys. Rev. Lett. **85**, 240 (2000) [hep-th/0004014]; U. Gunther and A. Zhuk, Class. Quant. Grav. **18**, 1441 (2001) [hep-ph/0006283]; M. Giovannini

Phys. Rev. **D63**, 064011 (2001) [hep-th/0011153]; J.E. Kim, B. Kyae and H.M. Lee, Phys. Rev. **D64**, 065011 (2001) [hep-th/0104150]; P. Kanti, R. Madden and K. Olive, Phys. Rev. **D64**, 044021 (2001) [hep-th/0104177]; R. Gregory and A. Padilla, Phys. Rev. **D65**, 084013 (2002) [hep-th/0104262]; F. Leblond, Phys. Rev. **D64**, 045016 (2001) [hep-ph/0104273]; S. Carroll and L. Mersini, Phys. Rev. **D64**, 124008 (2001) [hep-th/0105007]; F. Leblond, R. Myers and D. Winters, JHEP **0107**, 031 (2001) [hep-th/0106140]; R. Gregory and A. Padilla, Class. Quant. Grav. **19**, 279 (2002) [hep-th/0107108]; O. Corradini, A. Iglesias, Z. Kakushadze and P. Langfelder, Phys. Lett. **B521**, 96 (2001) [hep-th/0108055]; M. Giovannini and H. B. Meyer, Phys. Rev. **D64**, 124025 (2001) [hep-th/0108156]; C. Burgess, J. Cline, N. Constable and H. Firouzjahi, JHEP **0201**, 014 (2002) [hep-th/0112047]; S. Hayakawa, D. Ida, T. Shiromizu and T. Tanaka, Prog. Theor. Phys. Suppl. **148**, 128 (2002); S. Carroll and M. Guica, hep-th/0302067; I. Navarro, JCAP **0309**, 004 (2003) [hep-th/0302129]; Y. Aghababaie, C. Burgess, S. Parameswaran and F. Quevedo, Nucl. Phys. **B680**, 389 (2004) [hep-th/0304256]; I. Navarro, Class. Quant. Grav. **20**, 3603 (2003) [hep-th/0305014]; G. Gibbons, R. Guven and C. Pope, Phys. Lett. **B595**, 498 (2004) [hep-th/0307238]; Y. Aghababaie, C. Burgess, J. Cline, H. Firouzjahi, S. Parameswaran, F. Quevedo, G. Tasinato and I. Zavala, JHEP **0309**, 037 (2003) [hep-th/0308064]; H-P. Nilles, A. Papazoglou and G. Tasinato, Nucl. Phys. **B677**, 405 (2004) [hep-th/0309042]; H.M. Lee, Phys. Lett. **B587**, 117 (2004) [hep-th/0309050]; J-M. Schwindt and C. Wetterich, Phys. Lett. **B578**, 409 (2004) [hep-th/0309065]; B. de Carlos and J. Moreno, JHEP **0311**, 040 (2003) [hep-th/0309259]; R. Battye, B. Carter and A. Mennim, Phys. Rev. Lett. **92**, 201305 (2004) [hep-th/0312198]; H.M. Lee and G. Tasinato, JCAP **0404**, 009 (2004) [hep-th/0401221]; C. Burgess, Ann. Phys. **313**, 283 (2004) [hep-th/0402200]; I. Navarro and J. Santiago, JHEP **0404**, 062 (2004) [hep-th/0402204]; A. Maroto, Phys. Rev. **D69**, 101304 (2004) [hep-ph/0402278]; M. Graesser, J. Kile and P. Wang, Phys. Rev. **D70**, 024008 (2004) [hep-th/0403074]; N. Kaloper, JHEP **0405**, 061 (2004) [hep-th/0403208]; S. Kanno and J. Soda, JCAP **0407**, 002 (2004) [hep-th/0404207]; A. Kehagias, Phys. Lett. **B600**, 133 (2004) [hep-th/0406025]; J. Vinet and J. Cline, Phys. Rev. **D70**, 083514 (2004) [hep-th/0406141]; J. Garriga and M. Porrati, JHEP **0408**, 028 (2004) [hep-th/0406158]; P. Wang and X-H. Meng, hep-th/0406170; S. Randjbar-Daemi and V. Rubakov, JHEP **0410**, 054 (2004) [hep-th/0407176]; H.M. Lee and A. Papazoglou, hep-th/0407208; V. Nair and S. Randjbar-Daemi, hep-th/0408063; J. Vinet, hep-th/0408082; Y. Brihaye and B. Hartmann, hep-th/0408099; C. Burgess, F. Quevedo, G. Tasinato and I. Zavala, JHEP **0411**, 069 (2004) [hep-th/0408109]; S. Kanno and J. Soda, gr-qc/0410067; C. Burgess, hep-th/0411140; I. Navarro and J. Santiago, hep-th/0411250.

[2] J. Cline, J. Descheneau, M. Giovannini and J. Vinet, JHEP **0306**, 048 (2003) [hep-th/0304147].

[3] P. Bostock, R. Gregory, I. Navarro and J. Santiago, Phys. Rev. Lett. **92**, 221601 (2004) [hep-th/0311074].

[4] S. Davis, Phys. Rev. **D67**, 024030 (2003) [hep-th/0208205]; E. Gravannis and S. Willison, Phys. Lett. **B562**, 118 (2003) [hep-th/0209076]; S. Nojiri and S. Odintsov, JHEP **0007**, 049 (2000) [hep-th/0006232]; N. Mavromatos and J. Rizos, Phys. Rev. **D62**, 124004 (2000) [hep-th/0008074]; Y. Cho, I. Neupane and P. Wesson, Nucl. Phys. **B621**, 388 (2002) [hep-th/0104227].

[5] R. Capovilla and J. Guven, Phys. Rev. **D51**, 6736 (1995) [gr-qc/9411060]; Phys. Rev. **D52**, 1072 (1995) [gr-qc/9411061].

[6] B. Carter, J. Geom. Phys. **8**, 53 (1992); hep-th/9705172; gr-qc/0012036.

[7] Analogous is our convention for the ordering of indices in the definition of the other Riemann tensors.